

## APPENDIX 3

*“Everything that happens, happens as it should, and if you observe carefully, you will find this to be so”.*

Marcus Aurelius Antoninus (180 – 121 B.C.)

### Revolution of neutral body in a closed ellipse

The force of attraction  $F$  between two bodies of masses  $M$  and  $m$ , distance  $r$  apart in space, is given by Newton’s universal law of gravitation:

$$\mathbf{F} = \frac{-GMm}{r^2} \hat{\mathbf{u}} \quad (\text{A3.1})$$

where  $G$  is the gravitational constant and  $\hat{\mathbf{u}}$  is a unit vector in the radial direction. This force is extremely feeble that it is only noticeable in respect of huge masses like the planets and the Sun.

Two bodies under mutual attraction will revolve round their centre of mass with the larger mass  $M$  revolving in an inner orbit and the lighter mass  $m$  revolving in an outer orbit. The two bodies will revolve with the same angular velocity under equal and opposite forces of attraction. If  $M$  is very much larger than  $m$  as in the case of the Sun ( $M = 2 \times 10^{30} \text{ kg}$ ) and the Earth ( $6 \times 10^{24} \text{ kg}$ ), the Sun may be considered as almost stationary at a point while the Earth revolves at a distance  $r$  from the point (the centre of the Sun).

The gravitational force of attraction, on a moving body, is independent of its speed in a gravitational field. Newton’s second law of motion, on a body of mass  $m$  moving at time  $t$  with speed  $v$  and acceleration  $dv/dt$  in the radial direction, gives:

$$\mathbf{F} = \frac{-GMm}{r^2} \hat{\mathbf{u}} = m \frac{dv}{dt} \hat{\mathbf{u}} \quad (\text{A3.2})$$

The centripetal acceleration on a body revolving through angle  $\varphi$ , under a central force, gives the accelerating force as:

$$\mathbf{F} = \frac{-GMm}{r^2} \hat{\mathbf{u}} = m \left\{ \frac{d^2 r}{dt^2} - r \left( \frac{d\varphi}{dt} \right)^2 \right\} \hat{\mathbf{u}}$$

$$\frac{-GM}{r^2} = \frac{d^2r}{dt^2} - r \left( \frac{d\phi}{dt} \right)^2 \quad (\text{A3.3})$$

This is a differential equation of motion in  $r$  and  $\phi$  as functions of time  $t$ . We can reduce it to an equation of  $r$  as a function of  $\phi$ .

The angular momentum  $K$  of  $m$ , being a constant (Kepler's second law), gives:

$$K = mr^2 \frac{d\phi}{dt} \quad (\text{A3.4})$$

Making the substitution  $r = 1/u$  and with equation (A3.4), gives:

$$\frac{dr}{dt} = \frac{dr}{du} \frac{d\phi}{dt} \frac{du}{d\phi} = \frac{-K}{m} \frac{du}{d\phi} \quad (\text{A3.5})$$

$$\frac{d^2r}{dt^2} = \frac{d}{dt} \frac{dr}{dt} = \frac{d\phi}{dt} \frac{d}{d\phi} \left( \frac{-K}{m} \frac{du}{d\phi} \right) = \frac{-K^2 u^2}{m^2} \frac{d^2u}{d\phi^2} \quad (\text{A3.6})$$

Substituting equations (A3.6) and (A3.5) in equation (A3.3) gives:

$$\begin{aligned} -\frac{K^2 u^2}{m^2} \frac{d^2u}{d\phi^2} - \frac{K^2 u^3}{m^2} &= -GMu^2 \\ \frac{d^2u}{d\phi^2} + u &= \frac{GMm^2}{K^2} \end{aligned} \quad (\text{A3.7})$$

It should be noted that in equation (A3.7), the radiation component containing  $du/d\phi$ , in the case of a charged particle revolving under a central force, is missing.

Equation (A3.7) is a second order differential equation with constant coefficients. Trying  $u = A \exp(z\phi)$  as a solution, we obtain:

$$z^2 + 1 = 0$$

$$z^2 = \sqrt{-1} = \pm j$$

The general solution of equation (A3.7) is:

$$u = \frac{1}{r} = A \exp(j\varphi) + \frac{GMm^2}{K^2} \quad (\text{A3.8})$$

An appropriate solution is:

$$\frac{1}{r} = A \cos(\varphi + \beta) + \frac{GMm^2}{K^2} \quad (\text{A3.9})$$

where the amplitude  $A$  and phase angle  $\beta$  are determined from the initial conditions. If  $\beta = 0$ , equation (A3.9) may be written as:

$$\frac{1}{r} = \frac{GMm^2}{K^2} \left( 1 + \frac{AK^2}{GMm^2} \cos \varphi \right) \quad (\text{A3.10})$$

$$\frac{1}{r} = B \left( 1 + \frac{A}{B} \cos \varphi \right) = B(1 + \varepsilon \cos \varphi) \quad (\text{A3.11})$$

where  $B = GMm^2/K^2$ . Equation (A3.11) gives an ellipse, in the polar coordinates, with eccentricity  $\eta = A/B$ . The ellipse is shown as  $XYZW$  in Figure A3.1.

In Figure A3.1, the lighter mass  $m$  revolves, in angle  $\varphi$ , at a point  $P$ , in a closed ellipse, at a distance  $r$  from a much heavier mass  $M$  at one focus  $F_1$ . The other focus  $F_2$  is at a distance  $s$  from  $P$ . A property of an ellipse is that the distances  $s + r = 2a$ , the length of the major axis  $ZX$ . Other properties are obtained as the angle  $\varphi$  takes values  $0$ ,  $\pi/2$  and  $\pi$  radians. The line  $ZX$  is the major axis,  $WY$  is the minor axis and the chords  $CD$  and  $EG$  are the latus rectums, with half length  $l = a(1 - \eta^2) = l/B$ . The eccentricity of the ellipse is ratio of the distance between the foci ( $F_1F_2$ ) to the length of the major axis ( $ZX$ ). For a circle, the two foci coincide and the eccentricity  $\eta$  is equal to zero.

In planetary motion,  $X$ , the point of closest approach to the Sun, is called the perihelion and  $Z$ , the point of farthest separation, is the aphelion. A planet moves faster as it approaches the Sun; it is fastest at the perihelion and slowest at the aphelion. At any time, a planet moves in such a way that the difference in kinetic energy between two points is equal to the difference in potential energy. The reader should show that change in kinetic energy from the perihelion to the aphelion is the same as change in potential energy.

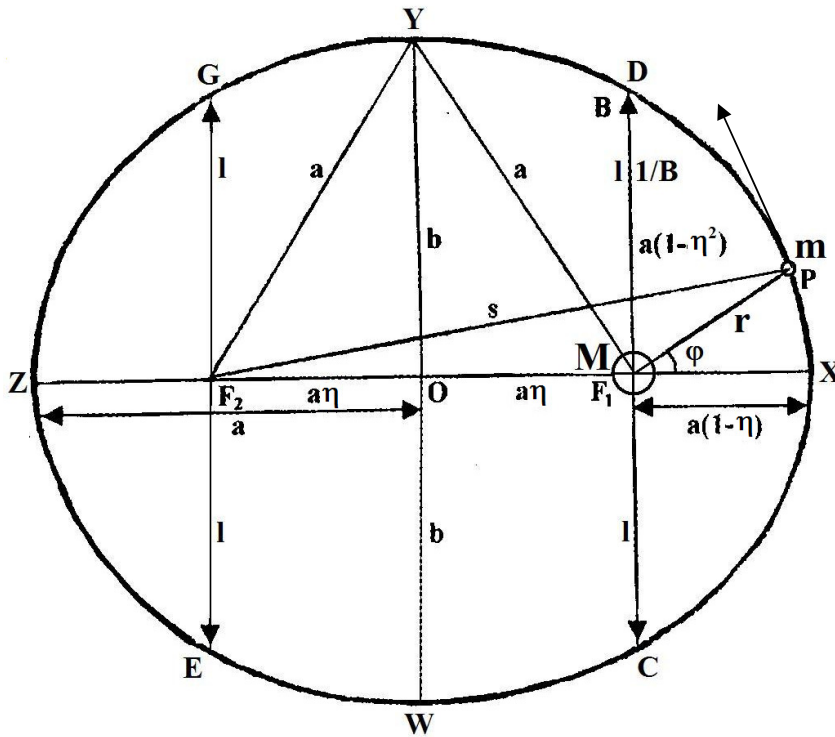


Figure A3.1 A body of lighter mass  $m$  revolving in an elliptic orbit, of eccentricity  $\eta$ , round a much heavier body of mass  $M$  at the centre of mass, the focus  $F_1$ ,

### Period of revolution in closed ellipse

The period of revolution  $T_e$  is obtained from equation (A3.4) as the definite integral:

$$T_e = \int_0^{2\pi} \frac{mr^2}{K} (d\phi)$$

Substituting for  $r$  from equation (A3.11), we obtain:

$$T_e = \frac{m}{KB^2} \int_0^{2\pi} (1 + \eta \cos \phi)^{-2} (d\phi)$$

Expanding the integral by the binomial theorem, gives:

$$T_e = \frac{m}{KB^2} \sum_{p=0}^{\infty} (-1)^p (1+p) \eta^p \int_0^{2\pi} \cos^p \varphi (d\varphi) = \frac{m}{KB^2} \sum_{p=0}^{\infty} Q_p$$

where  $p$  is an integer taking the values 0, 1, 2, 3... $\infty$ . As the terms for all odd values of  $p$  are zero, we shall take only the even  $p$ 's to give the equation:

$$Q_{2p} = (1+2p) \eta^{2p} \int_0^{2\pi} \cos^{2p} \varphi (d\varphi)$$

$$Q_0 = \int_0^{2\pi} (d\varphi) = 2\pi \quad (\text{A3.12})$$

$$Q_2 = 3\eta^2 \int_0^{2\pi} \cos^2 \varphi (d\varphi)$$

$$Q_2 = \frac{3\eta^2}{2} \int_0^{2\pi} (1 + \cos 2\varphi) (d\varphi) = \frac{3\eta^2}{2} (2\pi) \quad (\text{A3.13})$$

$$Q_4 = 5\eta^4 \int_0^{2\pi} \cos^4 \varphi (d\varphi)$$

$$Q_4 = \frac{5\eta^4}{8} \int_0^{2\pi} \{3 + \cos(4\varphi) + 4\cos(2\varphi)\} (d\varphi)$$

$Q_4$  is obtained as:

$$Q_4 = \frac{15\eta^4}{8} (2\pi) \quad (\text{A3.14})$$

$$T_e = \frac{m}{KB^2} \sum_{p=0}^{\infty} Q_{2p} = \frac{m}{KB^2} (Q_0 + Q_2 + Q_4 + Q_6 + \dots + Q_{\infty})$$

$$T_e = \frac{2\pi m}{KB^2} \left\{ 1 + \frac{3\eta^2}{2} + \frac{15\eta^4}{8} + \frac{35\eta^6}{16} + \dots + (-1)^p \frac{\left(\frac{-3}{2}\right)!}{\left(\frac{-3}{2} - p\right)! p!} \eta^{2p} \right\} \quad (\text{A3.15})$$

Here, the definition of factorial ( ! ) of a number is extended to include negative fractions. An inspection of equation (A3.15) reveals that it is identical to:

$$T_e = \frac{2\pi m}{KB^2} (1 - \eta^2)^{-\frac{3}{2}} \quad (\text{A3.16})$$

The period of revolution can also be obtained from equation (A3.4) and the area S of the ellipse. This is Kepler's second law:

$$\frac{dS}{dt} = \frac{S}{T_c} = \frac{r^2(d\varphi)}{2(dt)} = \frac{K}{2m}$$

$$T_e = \frac{2mS}{K} \quad (\text{A3.17})$$

Comparing equations (A3.17) and (A3.16) the area S is obtained as:

$$S = \frac{\pi}{B^2} (1 - \eta^2)^{-\frac{3}{2}} = \pi ab \quad (\text{A3.18})$$

It is left as an exercise to the reader to show, from the geometry of an ellipse, Figure A3.1, that the product  $ab = a^2(1 - \eta^2)^{1/2}$  and  $\pi ab$  is the area of an ellipse of eccentricity  $\eta$ , semi major axis  $a$  and semi minor axis  $b$ .